

# UNCONDITIONAL STRUCTURES OF TRANSLATES FOR $L_p(\mathbb{R}^d)$

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ABSTRACT. We prove that a sequence  $(f_i)_{i=1}^\infty$  of translates of a fixed  $f \in L_p(\mathbb{R})$  cannot be an unconditional basis of  $L_p(\mathbb{R})$  for any  $1 \leq p < \infty$ . In contrast to this, for every  $2 < p < \infty$ ,  $d \in \mathbb{N}$  and unbounded sequence  $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$  we establish the existence of a function  $f \in L_p(\mathbb{R}^d)$  and sequence  $(g_n^*)_{n \in \mathbb{N}} \subset L_p^*(\mathbb{R}^d)$  such that  $(T_{\lambda_n} f, g_n^*)_{n \in \mathbb{N}}$  forms an unconditional Schauder frame for  $L_p(\mathbb{R}^d)$ . In particular, there exists a Schauder frame of integer translates for  $L_p(\mathbb{R})$  if (and only if)  $2 < p < \infty$ .

## 1. INTRODUCTION

If  $d \in \mathbb{N}$  and  $\lambda \in \mathbb{R}^d$ , the *translation operator*  $T_\lambda$  is defined by  $T_\lambda f(x) = f(x - \lambda)$  for all  $x \in \mathbb{R}^d$  and  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Note that for the case  $d = 1$  and  $\lambda > 0$ , the operator  $T_\lambda$  is simply translation by  $\lambda$  units to the right. Given  $1 \leq p < \infty$ ,  $f \in L_p(\mathbb{R})$ , and  $\Lambda \subset \mathbb{R}$ , the resulting space  $X_p(f, \Lambda) \equiv \overline{\text{span}\{T_\lambda f\}_{\lambda \in \Lambda}}$  and set  $\{T_\lambda f\}_{\lambda \in \Lambda}$  have been studied in a variety of contexts and in particular arise in the study of wavelets and Gabor frames [HSWW, CDH].

Some of the natural problems to consider when studying translations of a fixed function  $f$  relate to characterizing when can  $X_p(f, \Lambda) = L_p(\mathbb{R}^d)$  and when can  $\{T_\lambda f\}_{\lambda \in \Lambda}$  be ordered to form a coordinate system such as a (unconditional) Schauder basis or (unconditional) Schauder frame for  $L_p(\mathbb{R}^d)$ . For  $d = 1$ , the cases when  $\Lambda = \mathbb{Z}$  or  $\Lambda = \mathbb{N}$  are of particular interest. For  $1 \leq p \leq 2$ , a Fourier transform argument yields that there does not exist an  $f \in L_p(\mathbb{R})$  such that  $X_p(f, \mathbb{Z}) = L_p(\mathbb{R})$  [AO]. On the other hand, for all  $\{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \setminus \mathbb{Z}$  such that  $\lim_{n \rightarrow \pm\infty} |\lambda_n - n| = 0$ , there exists  $f \in L_2(\mathbb{R})$  such that  $X_2(f, \mathbb{Z}) = L_2(\mathbb{R})$  [O].

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The case  $2 < p < \infty$  is completely different, as for all  $2 < p < \infty$  there exists  $f \in L_p(\mathbb{R})$  such that  $X_p(f, \mathbb{Z}) = L_p(\mathbb{R})$  and, moreover,  $T_m f \notin X_p(f, \mathbb{Z} \setminus \{m\})$  for all  $m \in \mathbb{Z}$  [AO].

Suppose that  $f \in L_p(\mathbb{R})$  and that  $\{T_\lambda f : \lambda \in \Lambda\}$  is an unconditional basic sequence in  $L_p(\mathbb{R})$ . What can be said about  $X_p(f, \Lambda)$ ? If  $1 \leq p \leq 2$  then  $\{T_\lambda f : \lambda \in \Lambda\}$  must be equivalent to the unit vector basis of  $\ell_p$ . If  $2 < p \leq 4$ , then  $X_p(f, \Lambda)$  embeds into  $\ell_p$  but  $(T_\lambda f)_{\lambda \in \Lambda}$  need not be equivalent to the unit vector basis of  $\ell_p$ . These results were shown in [OSSZ] and imply in particular that for all  $1 \leq p \leq 4$  (with [CDH] for the case  $p = 2$ ), there is no function  $f \in L_p(\mathbb{R})$  and set  $\Lambda \subset \mathbb{R}$  so that  $(T_\lambda f : \lambda \in \Lambda)$  is an unconditional basis for  $L_p(\mathbb{R})$ . For  $4 < p < \infty$ , there exists  $f \in L_p(\mathbb{R})$  and  $\Lambda \subseteq \mathbb{N}$  such that  $(T_\lambda f)_{\lambda \in \Lambda}$  is an unconditional basic sequence and  $L_p(\mathbb{R})$  embeds isomorphically into  $X_p(f, \Lambda)$  [OSSZ]. In Section 3 we prove that if  $(T_\lambda f)_{\lambda \in \Lambda}$  is an unconditional basic sequence in  $L_p(\mathbb{R})$  with  $2 < p < \infty$  such that  $X_p(f, \Lambda)$  is complemented in  $L_p(\mathbb{R})$  then  $(T_\lambda f)_{\lambda \in \Lambda}$  must be equivalent to the unit vector basis of  $\ell_p$ . In particular,  $X_p(f, \Lambda) \neq L_p(\mathbb{R})$ . Thus, by filling the gap  $(4, \infty)$ , we have for all  $1 \leq p < \infty$ , that there is no function  $f \in L_p(\mathbb{R})$  and set  $\Lambda \subset \mathbb{R}$  so that  $(T_\lambda f : \lambda \in \Lambda)$  is an unconditional basis for  $L_p(\mathbb{R})$ .

A basic sequence  $(x_i)$  can uniquely represent every vector in its closed span in terms of an infinite series. We now drop the uniqueness requirement of the representation and consider frames formed by translating a single function. Though there exists  $\Lambda \subset \mathbb{R}$  and  $f \in L_2(\mathbb{R})$  such that  $X_2(f, \Lambda) = L_2(\mathbb{R})$ , there does not exist  $\Lambda \subset \mathbb{R}$  and  $f \in L_2(\mathbb{R})$  such that  $\{T_\lambda f\}_{\lambda \in \Lambda}$  is a Hilbert frame for  $L_2(\mathbb{R})$  [CDH]. In the case of translation only by natural numbers, for all  $\Lambda \subset \mathbb{N}$  and  $f \in L_2(\mathbb{R})$ , the sequence  $(T_\lambda f)_{\lambda \in \Lambda}$  is a Hilbert frame for  $X_2(f, \Lambda)$  if and only if it is a Riesz basis for  $X_2(f, \Lambda)$ , i.e.,  $(T_\lambda f)$  must be equivalent to the unit vector basis of  $\ell_2$  [CCK]. In Section 3 we provide some background on Schauder frames for Banach spaces and prove that there exists a function  $f \in L_p(\mathbb{R})$  and sequence  $(g_n^*)_{n \in \mathbb{N}} \subset L_p^*(\mathbb{R})$  such that  $(T_n f, g_n^*)_{n \in \mathbb{N}}$  forms an unconditional Schauder frame for  $L_p(\mathbb{R})$  if (and only if)  $2 < p < \infty$ . More generally, we prove that for every  $2 < p < \infty$ ,  $d \in \mathbb{N}$  and unbounded sequence  $(\lambda_n)_{n \in \mathbb{N}}$

in  $\mathbb{R}^d$ , there exists a function  $f \in L_p(\mathbb{R}^d)$  and sequence  $(g_n^*)_{n \in \mathbb{N}}$  such that  $(T_{\lambda_n} f, g_n^*)_{n \in \mathbb{N}}$  forms an unconditional Schauder frame for  $L_p(\mathbb{R}^d)$ .

For  $2 < p < \infty$ , if  $L_p$  embeds into  $X_p(f, \Lambda)$  and  $X_p(f, \Lambda)$  is complemented in  $L_p(\mathbb{R})$  then  $(T_\lambda f)_{\lambda \in \Lambda}$  cannot be an unconditional basic sequence in  $L_p(\mathbb{R})$ . However, it is possible that  $(T_\lambda f)_{\lambda \in \Lambda}$  can be blocked into an unconditional FDD. We prove in Section 4 that for  $2 < p < \infty$  there exists  $f \in L_p(\mathbb{R})$  and  $\Lambda \subseteq \mathbb{N}$  so that  $X_p(f, \Lambda)$  is isomorphic to  $L_p$ ,  $X_p(f, \Lambda)$  is complemented in  $L_p$ , and  $\{T_\lambda f\}_{\lambda \in \Lambda}$  can be blocked to form an unconditional finite dimensional decomposition (unconditional FDD) for  $X_p(f, \Lambda)$ .

In Section 5, we study the restriction operator  $T_I : L_p \rightarrow L_p$  given by  $x \mapsto x|_I$  where  $I \subset \mathbb{R}$  is some bounded interval. Assuming  $(T_{\lambda_i} f)$  is an unconditional basic sequence, we characterize for what values of  $1 \leq p < \infty$  must the map  $T_I : X_p(f, (\lambda_i)) \rightarrow L_p$  be compact for all bounded intervals  $I \subset \mathbb{R}$ . We prove as well other relationships between the restriction operator  $T_I : X_p(f, (\lambda_i)) \rightarrow L_p$  and the structure of  $X_p(f, (\lambda_i))$ . Lastly, in Section 6 we give some open problems.

## 2. UNCONDITIONAL BASES OF TRANSLATES

**Theorem 2.1.** *Let  $2 < p < \infty$  and  $f \in L_p(\mathbb{R})$ . If  $(T_\lambda f : \lambda \in \Lambda)$  is an unconditional basis of  $X_p(f, \Lambda)$  and  $X_p(f, \Lambda)$  is complemented in  $L_p(\mathbb{R})$ , then  $(T_\lambda f)_{\lambda \in \Lambda}$  is equivalent to the unit vector basis of  $\ell_p$ .*

We will need the following result from [JO].

**Proposition 2.2.** [JO, Section 3, Lemma 2]. *Let  $1 \leq q \leq 2$ . Let  $(f_i) \subseteq L_q(\mathbb{R})$  be semi-normalized and unconditional basic. Assume that for some  $\varepsilon > 0$  there exists a sequence of disjoint measurable sets  $(B_i)_{i=1}^\infty$  with  $\|f_i|_{B_i}\|_q \geq \varepsilon$ , for all  $i$ . Then  $(f_i)_{i=1}^\infty$  is equivalent to the unit vector basis of  $\ell_q$ .*

*Proof of Theorem 2.1.* Without loss of generality we can assume that  $\|T_{\lambda_i} f\|_p = \|f\|_p = 1$ , for  $i \in \mathbb{N}$ . Put  $f_i = T_{\lambda_i} f$  and  $X = X_p(f, \Lambda)$  and let  $Y \subset L_p(\mathbb{R})$  be a complement of  $X$  in  $L_p(\mathbb{R})$ . Denote the biorthogonals of  $(f_i)$  inside  $X^*$  by  $(\bar{g}_i)$  and let  $g_i$ ,  $i \in \mathbb{N}$ , be the extension of  $\bar{g}_i$  to an element of  $L_p^*(\mathbb{R}) = L_q(\mathbb{R})$  with  $g_i|_Y \equiv 0$ . Thus  $(g_i)$  is an unconditional basic sequence inside  $L_q(\mathbb{R})$  which is biorthogonal to  $(f_i)$  and vanishes on  $Y$ .

Recall that if  $\{T_\lambda f : \lambda \in \Lambda\}$  is a basic sequence in  $L_p(\mathbb{R})$ , for some  $f \in L_p(\mathbb{R})$  and  $\Lambda \subseteq \mathbb{R}$ , then  $\Lambda$  is uniformly discrete [OSSZ]. That is, we may choose  $\delta > 0$  such that

$$0 < \delta < \inf\{|\lambda - \mu| : \lambda, \mu \in \Lambda, \lambda \neq \mu\}.$$

For  $j \in \mathbb{Z}$ , we define the interval  $I_j = [j\delta, (j+1)\delta)$ .

**Claim.** There exist  $N \in \mathbb{N}$  and  $\varepsilon > 0$ , so that: for all  $i \in \mathbb{N}$  there is an  $l_i \in \mathbb{Z}$  and a  $j_i \in \{l_i, l_i + 1, \dots, l_i + N\}$ , so that

$$l_i \neq l_{i'}, \text{ if } i \neq i', \text{ and } \|g_i|_{I_{j_i}}\|_q > \varepsilon.$$

Indeed, choose first  $l_0 \in \mathbb{Z}$  and  $N \in \mathbb{N}$  so that

$$\|f\|_{\mathbb{R} \setminus \bigcup_{j=l_0}^{l_0+N-1} I_j}^p = \int_{-\infty}^{l_0\delta} |f(z)|^p dz + \int_{(l_0+N)\delta}^{\infty} |f(z)|^p dz < \frac{1}{2^p} \left( \sup_{i \in \mathbb{N}} \|g_i\|_q^p \right)^{-1}$$

Then for  $i \in \mathbb{N}$  choose  $l_i \in \mathbb{Z}$  such that

$$l_0\delta \leq (l_i + 1)\delta - \lambda_i < (l_0 + 1)\delta.$$

Note that if  $i \neq i'$  it follows that  $|\lambda_i - \lambda_{i'}| > \delta$  and, thus,  $l_i \neq l_{i'}$ . Moreover,

$$\begin{aligned} \|f_i\|_{\mathbb{R} \setminus \bigcup_{j=l_i}^{l_i+N} I_j}^p &= \int_{-\infty}^{l_i\delta} |f(x - \lambda_i)|^p dx + \int_{(l_i+N+1)\delta}^{\infty} |f(x - \lambda_i)|^p dx \\ &= \int_{-\infty}^{l_i\delta - \lambda_i} |f(z)|^p dz + \int_{(l_i+N+1)\delta - \lambda_i}^{\infty} |f(z)|^p dz \\ &\leq \int_{-\infty}^{l_0\delta} |f(z)|^p dz + \int_{(l_0+N)\delta}^{\infty} |f(z)|^p dz < \frac{1}{2^p} \left( \sup_{i \in \mathbb{N}} \|g_i\|_q^p \right)^{-1}. \end{aligned}$$

Thus, by Hölder's Theorem and the fact that  $\|f\|_p = 1$ , it follows that

$$\|g_i|_{\bigcup_{j=l_i}^{l_i+N} I_j}\|_q \geq \int_{\bigcup_{j=l_i}^{l_i+N} I_j} g_i f_i dz = 1 - \int_{\mathbb{R} \setminus \bigcup_{j=l_i}^{l_i+N} I_j} g_i f_i \geq 1 - \|g_i\|_q \|f|_{\mathbb{R} \setminus \bigcup_{j=l_i}^{l_i+N} I_j}\|_p \geq \frac{1}{2}.$$

Letting  $\varepsilon = \frac{1}{2(N+1)}$  we deduce our claim.

Since the  $l_i$ 's are distinct, it follows that for each  $k \in \mathbb{Z}$

$$\#\{i \in \mathbb{N} : j_i = k\} \leq \#\{i : k \in [l_i, l_i + N]\} = \#\{i : l_i \in [k - N, k]\} \leq N + 1.$$

For each  $k \in \mathbb{Z}$  we can order the (possibly empty) set  $\{i \in \mathbb{N} : j_i = k\}$  into  $i(k, 1), i(k, 2), \dots, i(k, m_k)$ , with  $0 \leq m_k \leq N + 1$  (where we let  $m_k = 0$  if  $\{i \in \mathbb{N} : j_i = k\}$  is empty).

For  $k \in \mathbb{Z}$  and  $s \in \{1, 2, \dots, m_k\}$  put  $g_k^{(s)} = g_{i(k,s)}$ .

For each  $s \leq N + 1$  it follows that the sequence  $(g_k^{(s)} : k \in \mathbb{Z} \text{ and } m_k \geq s)$ , satisfies the condition of Proposition 2.2, with  $B_k = [k\delta, (k+1)\delta)$ , as long it is infinite and must therefore be equivalent to the unit vector basis of  $\ell_q$ . Thus, since  $\overline{\text{span}\{g_i\}_{i \in \mathbb{N}}}$  is the unconditional sum of  $[g_k^{(s)} : k \in \mathbb{Z} \text{ and } m_k \geq s]$ ,  $s = 1, 2, \dots, N + 1$ , it follows that  $(g_i)$  must be equivalent to the unit vector basis of  $\ell_q$ . Since  $X$  is complemented in  $L_p(\mathbb{R})$  and  $g_i|_Y = 0$  (recall that  $L_p = X \oplus Y$ ) and  $g_i|_X = \bar{g}_i$  for  $i \in \mathbb{N}$  it follows that  $(\bar{g}_i)$  is equivalent to  $(g_i)$  and, thus, also equivalent to the unit vector basis of  $\ell_q$ . But this implies that  $(f_i)$  is equivalent to the unit vector basis of  $\ell_p$ .  $\square$

**Corollary 2.3.** *If  $f \in L_p(\mathbb{R})$ ,  $1 \leq p < \infty$ ,  $(T_\lambda f)_{\lambda \in \Lambda}$  is an unconditional basis for  $X_p(f, \lambda)$  then  $X_p(f, \Lambda) \neq L_p(\mathbb{R})$ .*

### 3. UNCONDITIONAL SCHAUDER FRAMES OF TRANSLATES

In Section 2, it was shown that there does not exist an unconditional basis of translates of a single function for  $L_p(\mathbb{R})$  for any value  $1 \leq p < \infty$ . In contrast to this, we will show that  $L_p(\mathbb{R})$  has an unconditional Schauder frame of integer translates of a single function if

(and only if)  $2 < p < \infty$ . Before proving this result, we will develop some basic theory of Schauder frames.

If  $X$  is a separable Banach space, then a sequence  $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$  is called a *Schauder frame* for  $X$  if

$$(1) \quad x = \sum_{i=1}^{\infty} g_i^*(x)x_i \quad \text{for all } x \in X.$$

A Schauder frame  $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$  is called an *unconditional Schauder frame* for  $X$  if the series (1) converges unconditionally for all  $x \in X$ . Recall that a series converges *unconditionally* if it converges for any ordering of the elements of the series.

Let  $X$  be a separable Banach space. Assume that a sequence  $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$  satisfies that the operator  $S : X \rightarrow X$  defined by  $S(x) = \sum_{i=1}^{\infty} g_i^*(x)x_i$  is well defined (and hence bounded due to the uniform boundedness principle).  $S$  is called the *frame operator* for  $(x_i, g_i^*)_{i=1}^\infty$ . Note that the sequence  $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$  is a Schauder frame if and only if the frame operator is the identity. We define  $(x_i, g_i^*)_{i=1}^\infty$  to be an *approximate Schauder frame* if the frame operator is bounded, one to one, and onto (hence has bounded inverse), and we define  $(x_i, g_i^*)_{i=1}^\infty$  to be an *unconditional approximate Schauder frame* if it is an approximate Schauder frame and the series  $\sum_{i=1}^{\infty} g_i^*(x)x_i$  converges unconditionally for all  $x \in X$ . A similar notion of a frame was studied by Thomas in the context of  $\ell_\infty^n$  [T].

**Lemma 3.1.** *Let  $X$  be a separable Banach space and let  $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$  be an approximate Schauder frame for  $X$  with frame operator  $S$ . Then  $(x_i, (S^{-1})^*g_i^*)_{i=1}^\infty$  is a Schauder frame for  $X$ . Furthermore, if  $(x_i, g_i^*)_{i=1}^\infty$  is an unconditional approximate Schauder frame for  $X$ , then  $(x_i, (S^{-1})^*g_i^*)_{i=1}^\infty$  is an unconditional Schauder frame for  $X$ .*

*Proof.* Let  $x \in X$ . We have that  $S$  and  $S^{-1}$  are bounded. Thus,

$$x = S(S^{-1}x) = \sum_{i=1}^{\infty} g_i^*(S^{-1}x)x_i = \sum_{i=1}^{\infty} ((S^{-1})^*g_i^*)(x)x_i.$$

Hence,  $(x_i, (S^{-1})^* g_i^*)_{i=1}^\infty \subset X \times X^*$  is a Schauder frame for  $X$ . Assume that  $(x_i, g_i^*)_{i=1}^\infty$  is an unconditional approximate Schauder frame for  $X$ . If  $x \in X$  and  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  is a permutation, then  $S(y) = \sum_{i=1}^\infty g_{\pi(i)}^*(y) x_{\pi(i)}$  for all  $y \in X$ . Hence,

$$x = S(S^{-1}x) = \sum_{i=1}^\infty g_{\pi(i)}^*(S^{-1}x) x_{\pi(i)} = \sum_{i=1}^\infty ((S^{-1})^* g_{\pi(i)}^*)(x) x_{\pi(i)} .$$

□

In particular, Lemma 3.1 implies that  $L_p(\mathbb{R}^d)$  has a (unconditional) Schauder frame formed by translating a single function if and only if it has an (unconditional) approximate Schauder frame formed by translating a single function. This is important for us, as we will provide an explicit construction for an approximate Schauder frame of translates for  $L_p(\mathbb{R}^d)$  and then apply Lemma 3.1 to obtain a Schauder frame of translates for  $L_p(\mathbb{R}^d)$  for any  $p > 2$ . One way to verify that a sequence  $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$  with frame operator  $S$  is an approximate Schauder frame is to show that  $\|S - Id_X\| < 1$ .

**Theorem 3.2.** *Let  $2 < p < \infty$  and  $d \in \mathbb{N}$ . If  $(\lambda_n)_{n \in \mathbb{N}}$  is an unbounded sequence in  $\mathbb{R}^d$  then there exists a function  $f \in L_p(\mathbb{R}^d)$  and a sequence  $(g_n^*)_{n \in \mathbb{N}} \subset L_p^*(\mathbb{R}^d)$  such that  $(T_{\lambda_n} f, g_n^*)_{n \in \mathbb{N}}$  forms an unconditional Schauder frame for  $L_p(\mathbb{R}^d)$ .*

*Proof.* Let  $(e_i)_{i=1}^\infty$  be a normalized unconditional Schauder basis for  $L_p(\mathbb{R}^d)$  with biorthogonal functionals  $(e_i^*)_{i=1}^\infty$  such that  $e_i \in L_p(\mathbb{R}^d)$  is a function satisfying  $\text{diam}(\text{supp}(e_i)) \leq 1$  for all  $i \in \mathbb{N}$ . Let  $C_u$  be the constant of unconditionality of  $(e_i)_{i=1}^\infty$ . For each  $k \in \mathbb{N}$ , choose  $N_k \in \mathbb{N}$  such that  $(\sum_{k=1}^\infty N_k^{1-p/2})^{1/p} < \frac{1}{2C_u}$ . We now inductively construct natural numbers  $n_{1,1} < n_{1,2} < \dots < n_{1,N_1} < n_{2,1} < \dots < n_{2,N_2} < \dots$  such that if  $(k, i) > (s, t)$ , in the lexicographic order, then

$$(2) \quad |\lambda_{n_{k,i}} - \lambda_{n_{s,t}}| > 1, \text{ and}$$

$$(3) \quad \text{supp}(T_{-\lambda_{n_{k,i}}} e_k) \cap \text{supp}(T_{-\lambda_{n_{s,t}}} e_s) = \emptyset ,$$

and if  $(k, i) > (s, t)$ ,  $(k, i) \geq (k', i')$ ,  $(k, i) \geq (s', t')$ ,  $(s, t) \neq (s', t')$  and  $(s', t') \neq (k', i')$  then

$$(4) \quad \text{supp}(T_{\lambda_{n_{s,t}} - \lambda_{n_{k,i}}} e_k) \cap \text{supp}(T_{\lambda_{n_{s',t'}} - \lambda_{n_{k',i'}}} e_{k'}) = \emptyset,$$

and, finally, if  $(k, i) > (s, t)$ ,  $(k, i) \geq (k', i')$ ,  $(k, i) > (s', t')$  and  $(s', t') \neq (k', i')$  then

$$(5) \quad \text{supp}(T_{\lambda_{n_{k,i}} - \lambda_{n_{s,t}}} e_s) \cap \text{supp}(T_{\lambda_{n_{s',t'}} - \lambda_{n_{k',i'}}} e_{k'}) = \emptyset.$$

As we are choosing  $n_{k,i} \in \mathbb{N}$ , we can clearly satisfy (2) and (3) by making  $n_{k,i}$  sufficiently large since each of the functions  $e_j$  has compact support, and the sequence  $(\lambda_n)_{n \in \mathbb{N}}$  is unbounded. As we have  $\text{diam}(\text{supp}(e_i)) \leq 1$  for all  $i \in \mathbb{N}$ , we will automatically satisfy (4) when  $(k, i) = (k', i')$ . The (finitely many) remaining cases of (4) and (5) can then be satisfied by making  $n_{k,i}$  larger still using again the assumptions that the functions  $e_j$  have compact support, and the sequence  $(\lambda_n)_{n \in \mathbb{N}}$  is unbounded.

As a result of the above inductive construction, we obtain the following two conditions. For all  $(k, i) \neq (s, t)$ ,

$$(6) \quad \text{supp}(T_{-\lambda_{n_{k,i}}} e_k) \cap \text{supp}(T_{-\lambda_{n_{s,t}}} e_s) = \emptyset.$$

For all  $(s, t), (s', t'), (k, i), (k', i')$ , if  $(s, t) \neq (s', t')$ ,  $(s, t) \neq (k, i)$  and  $(s', t') \neq (k', i')$ , then

$$(7) \quad \text{supp}(T_{\lambda_{n_{s,t}} - \lambda_{n_{k,i}}} e_k) \cap \text{supp}(T_{\lambda_{n_{s',t'}} - \lambda_{n_{k',i'}}} e_{k'}) = \emptyset.$$

Set  $f := \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-1/2} T_{-\lambda_{n_{k,i}}} e_k$ . Our first step is to show that  $f \in L_p(\mathbb{R}^d)$ .

$$\begin{aligned} \int |f|^p d\mu &= \int \left| \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-1/2} T_{-\lambda_{n_{k,i}}} e_k \right|^p d\mu \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-p/2} \int |e_k|^p d\mu \quad \text{by (6)} \\ &= \sum_{k=1}^{\infty} N_k^{1-p/2} \quad \text{as } \|e_k\| = 1 \text{ for all } k \in \mathbb{N} \\ &< \frac{1}{2^p C_u^p}. \end{aligned}$$



Thus we have that  $f \in L_p(\mathbb{R}^d)$ . For each  $j \in \mathbb{N}$ , we define  $g_j^* \in L_p^*(\mathbb{R}^d)$  by

$$g_j^* = \begin{cases} N_k^{-1/2} e_k^* & \text{if } j = n_{k,i} \text{ for some } k \in \mathbb{N} \text{ and } 1 \leq i \leq N_k, \\ 0 & \text{otherwise.} \end{cases}$$

We now show that  $\sum_{i=1}^{\infty} g_i^*(h) T_{\lambda_i} f$  converges unconditionally for all  $h \in L_p(\mathbb{R}^d)$ . In particular, this would imply that the frame operator for  $(T_{\lambda_n} f, g_n^*)_{n \in \mathbb{N}}$  would be well defined and bounded. By Proposition 1.c.1 in [LT], to prove that a series  $\sum_{i \in \mathbb{N}} x_i$  in a Banach space  $X$  converges unconditionally, it is sufficient to prove that for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all finite subsets  $A \subset \mathbb{N}$  with  $\min(A) > N$ ,  $\|\sum_{i \in A} x_i\| < \varepsilon$ . Let  $\varepsilon > 0$  and let  $h \in L_p(\mathbb{R}^d)$  such that  $\|h\| = 1$ . Choose  $M \in \mathbb{N}$  such that  $\sum_{i=M}^{\infty} N_i^{1-p/2} < \varepsilon$  and  $\|\sum_{i=M}^{\infty} e_i^*(h) e_i\| < \varepsilon$ . Let  $A \subset \mathbb{N}$  such that  $\min(A) \geq n_{M,1}$ . We now have the following estimate.

$$\begin{aligned} \left\| \sum_{i \in A} g_i^*(h) T_{\lambda_i} f \right\| &= \left\| \sum_{\substack{(s,t) \in \mathbb{N}^2 \\ n_{s,t} \in A}} N_s^{-1/2} e_s^*(h) \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-1/2} T_{\lambda_{n_{s,t}} - \lambda_{n_{k,i}}} e_k \right\| \\ &\leq \left\| \sum_{\substack{(s,t) \in \mathbb{N}^2 \\ n_{s,t} \in A}} e_s^*(h) N_s^{-1} e_s \right\| + \left\| \sum_{\substack{(s,t) \in \mathbb{N}^2 \\ n_{s,t} \in A}} N_s^{-1/2} e_s^*(h) \sum_{(k,i) \neq (s,t)} N_k^{-1/2} T_{\lambda_{n_{s,t}} - \lambda_{n_{k,i}}} e_k \right\| \\ &= \left\| \sum_{\substack{(s,t) \in \mathbb{N}^2 \\ n_{s,t} \in A}} e_s^*(h) N_s^{-1} e_s \right\| + \left( \sum_{\substack{(s,t) \in \mathbb{N}^2 \\ n_{s,t} \in A}} N_s^{-p/2} |e_s^*(h)|^p \sum_{(k,i) \neq (s,t)} N_k^{-p/2} \right)^{1/p} \quad \text{by (7)} \\ &\leq C_u \left\| \sum_{s=M}^{\infty} \sum_{t=1}^{N_s} e_s^*(h) N_s^{-1} e_s \right\| + C_u \left( \sum_{s=M}^{\infty} \sum_{t=1}^{N_s} N_s^{-p/2} \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-p/2} \right)^{1/p}, \\ &\quad \text{as } \min(A) \geq n_{M,1} \text{ and } |e_s^*(h)| \leq C_u \\ &= C_u \left\| \sum_{s=M}^{\infty} e_s^*(h) e_s \right\| + C_u \left( \sum_{s=M}^{\infty} N_s^{1-p/2} \sum_{k=1}^{\infty} N_k^{1-p/2} \right)^{1/p} < C_u \varepsilon + C_u \varepsilon^{1/p} \frac{1}{2C_u}. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, the series  $\sum_{i=1}^{\infty} g_i^*(h) T_i f$  converges unconditionally.

Let  $S$  be the frame operator for  $(T_{\lambda_n}f, g_n^*)_{n \in \mathbb{N}}$ . To show that  $(T_{\lambda_n}f, g_n^*)_{n \in \mathbb{N}}$  forms an approximate unconditional Schauder frame for  $L_p(\mathbb{R}^d)$  it suffices to show that  $\|S - Id_{L_p(\mathbb{R}^d)}\| < \frac{1}{2}$ . Let  $h \in L_p(\mathbb{R}^d)$  such that  $\|h\| = 1$ . Choose  $M \in \mathbb{N}$  such  $\|\sum_{i=M+1}^{\infty} e_i^*(h)e_i\| < \frac{1}{8}$  and  $\|\sum_{i=n_M, N_M+1}^{\infty} g_i^*(h)T_{\lambda_i}f\| < \frac{1}{8}$ . Then

$$\begin{aligned}
\|h - S(h)\| &= \left\| \sum_{i=1}^{\infty} e_i^*(h)e_i - \sum_{i=1}^{\infty} g_i^*(h)T_{\lambda_i}f \right\| \\
&< \left\| \sum_{i=1}^M e_i^*(h)e_i - \sum_{i=1}^{n_M, N_M} g_i^*(h)T_{\lambda_i}f \right\| + \frac{1}{8} + \frac{1}{8} \\
&= \left\| \sum_{i=1}^M e_i^*(h)e_i - \sum_{s=1}^M \sum_{t=1}^{N_s} N_s^{-1/2} e_s^*(h) \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-1/2} T_{\lambda_{n_s, t} - \lambda_{n_k, i}} e_k \right\| + \frac{1}{4} \\
&\leq \left\| \sum_{i=1}^M e_i^*(h)e_i - \sum_{s=1}^M \sum_{t=1}^{N_s} N_s^{-1} e_s^*(h)e_s \right\| \\
&\quad + \left\| \sum_{s=1}^M \sum_{t=1}^{N_s} N_s^{-1/2} e_s^*(h) \sum_{(k,i) \neq (s,t)} N_k^{-1/2} T_{\lambda_{n_s, t} - \lambda_{n_k, i}} e_k \right\| + \frac{1}{4} \\
&= \left\| \sum_{s=1}^M \sum_{t=1}^{N_s} N_s^{-1/2} e_s^*(h) \sum_{(k,i) \neq (s,t)} N_k^{-1/2} T_{\lambda_{n_s, t} - \lambda_{n_k, i}} e_k \right\| + \frac{1}{4} \\
&= \left( \sum_{s=1}^M \sum_{t=1}^{N_s} N_s^{-p/2} |e_s^*(h)|^p \sum_{(k,i) \neq (s,t)} N_k^{-p/2} \right)^{1/p} + \frac{1}{4} \quad \text{by (7)} \\
&\leq C_u \left( \sum_{s=1}^{\infty} \sum_{t=1}^{N_s} N_s^{-p/2} \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-p/2} \right)^{1/p} + \frac{1}{4} \\
&= C_u \left( \sum_{s=1}^{\infty} N_s^{1-p/2} \sum_{k=1}^{\infty} N_k^{1-p/2} \right)^{1/p} + \frac{1}{4} \\
&< C_u \frac{1}{2C_u} \frac{1}{2C_u} + \frac{1}{4} < \frac{1}{2}.
\end{aligned}$$

Thus  $\|S - Id_{L_p(\mathbb{R}^d)}\| < \frac{1}{2}$  and hence  $S$  is bounded and has a bounded inverse. This gives that  $(T_{\lambda_n}f, g_n^*)_{n \in \mathbb{N}}$  forms an approximate unconditional Schauder frame for  $L_p(\mathbb{R}^d)$ , and hence  $(T_{\lambda_n}f, (S^{-1})^*g_n^*)_{n \in \mathbb{N}}$  forms an unconditional Schauder frame for  $L_p(\mathbb{R}^d)$  by Lemma 3.1.  $\square$

We now discuss some consequences of Theorem 3.2. Given a Schauder frame  $(x_i, f_i)_{i=1}^\infty \subset X \times X^*$ , let  $H_n : X \rightarrow X$  be the operator  $H_n(x) = \sum_{i \geq n} f_i(x)x_i$ . The frame  $(x_i, f_i)_{i=1}^\infty$  is called *shrinking* if  $\|x^* \circ H_n\| \rightarrow 0$  for all  $x^* \in X^*$ . A Schauder frame  $(x_i, f_i)_{i=1}^\infty \subset X \times X^*$  for a Banach space  $X$  is shrinking if and only if  $(f_i, x_i)_{i=1}^\infty \subset X^* \times X^{**}$  is a Schauder frame for  $X^*$  [CL]. Furthermore, every unconditional Schauder frame for a reflexive Banach space is shrinking [CLS],[L]. Thus the following corollary of Theorem 3.2 ensues.

**Corollary 3.3.** *Let  $1 < q < 2$  and  $d \in \mathbb{N}$ . If  $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$  is unbounded then there exists a function  $f^* \in L_q^*(\mathbb{R}^d)$  and sequence  $(g_n)_{n \in \mathbb{N}} \subset L_q(\mathbb{R}^d)$  such that  $(g_n, T_{\lambda_n}f^*)_{n \in \mathbb{N}}$  forms a Schauder frame for  $L_q(\mathbb{R}^d)$ .*

Note that in Corollary 3.3, the dual functionals  $(T_{\lambda_n}f^*)_{n \in \mathbb{N}}$  are translations of a single function as opposed to the vectors  $(g_n)_{n \in \mathbb{N}}$ .

In [CDOSZ], it is proven that every Schauder frame has an associated basis. Essentially, that means that Schauder frames can be considered as projections of bases onto complemented subspaces. In [BFL], it is proven that every shrinking Schauder frame for a reflexive Banach space has a shrinking and boundedly complete associated basis, and it follows from the proof that if the frame is unconditional then the basis will be unconditional as well. Thus, the Schauder frame  $(T_{\lambda_n}f, g_n^*)_{n \in \mathbb{N}}$  for  $L_p(\mathbb{R}^d)$  will have an unconditional, shrinking, and boundedly complete associated basis.

#### 4. UNCONDITIONAL FDDs OF TRANSLATES

In Section 2, it was shown that for all  $f \in L_p(\mathbb{R})$ ,  $1 \leq p < \infty$ , and  $\Lambda \subset \mathbb{R}$ , if  $(T_\lambda f)_{\lambda \in \Lambda}$  is an unconditional basic sequence and  $X_p(f, \Lambda)$  is complemented in  $L_p(\mathbb{R})$  then  $(T_\lambda f)_{\lambda \in \Lambda}$

is equivalent to the unit vector basis for  $\ell_p$ . Instead of considering when  $(T_\lambda f)_{\lambda \in \Lambda}$  is an unconditional basic sequence, we now study the cases where  $(T_\lambda f)_{\lambda \in \Lambda}$  can be blocked into an unconditional FDD. Given a Banach space  $X$ , recall that a sequence of finite dimensional spaces  $(F_i)_{i=1}^\infty \subset X$  is called a *finite dimensional decomposition* or *FDD* for  $X$  if for every  $x \in X$  there exists for all  $i \in \mathbb{N}$  a unique  $x_i \in F_i$  such that  $x = \sum_{i=1}^\infty x_i$ . An FDD is called unconditional if the series  $x = \sum_{i=1}^\infty x_i$  converges unconditionally for all  $x \in X$ .

**Theorem 4.1.** *Let  $2 < p < \infty$ . There exists  $f \in L_p(\mathbb{R})$  and a subsequence  $(n_i)_{i=1}^\infty$  of  $\mathbb{N}$  so that for  $X = X_p(f, (-n_i)_{i=1}^\infty)$ , i)  $X$  is isomorphic to  $L_p(\mathbb{R})$ , ii)  $X$  is complemented in  $L_p(\mathbb{R})$ , and iii) there exists a partition of  $\mathbb{N}$  into successive intervals  $(J_j)_{j=1}^\infty$  so that setting  $F_j = \text{span}\{T_{-n_i}f\}_{i \in J_j}$ ,  $(F_j)_{j=1}^\infty$  forms an unconditional FDD for  $X$ .*

*Proof of Theorem 4.1.* Let  $\varepsilon > 0$  and choose a subsequence  $(N_i)_{i=1}^\infty$  of  $\mathbb{N}$  so that

$$(8) \quad \sum_{j=1}^\infty N_j^{1-\frac{p}{2}} < \infty \quad \text{and} \quad \sum_{j=1}^\infty N_j^{\frac{1}{p}-\frac{1}{2}} < \varepsilon .$$

Of course the second condition implies the first but we state both as they will be used.

Let  $(h_j^i)_{j=1}^\infty$  be the normalized Haar basis for  $L_p[3^i, 3^{i+1}]$  for  $i \in \mathbb{N}$ . Partition  $\mathbb{N}$  into successive intervals  $J_1, J_2, \dots$  so that  $|J_j| = N_j$  for  $j \in \mathbb{N}$ .

Let

$$f = \sum_{j=1}^\infty \sum_{i \in J_j} \frac{1}{\sqrt{N_j}} h_j^i \quad \text{and let} \quad f_i = T_{-3^i} f .$$

The choice of  $3^i$  above yields, as in Section 2, that for  $i \in J_j$ ,

$$(9) \quad f_i = \frac{1}{\sqrt{N_j}} h_j + g_i$$

where  $(h_j)$  is the normalized Haar basis for  $L_p[0, 1]$  and moreover the functions  $(g_i)$  have disjoint supports in  $\mathbb{R}$ . Indeed

$$\text{supp } g_i = \bigcup_{\ell \neq i} [3^\ell - 3^i, 3^\ell - 3^i + 1] .$$

$f \in L_p(\mathbb{R})$  since

$$\|f\|_p^p = \sum_{j=1}^{\infty} \sum_{i \in J_j} \left( \frac{1}{\sqrt{N_j}} \right)^p = \sum_{j=1}^{\infty} N_j \left( \frac{1}{\sqrt{N_j}} \right)^p = \sum_{j=1}^{\infty} N_j^{1-\frac{p}{2}} < \infty \text{ (by (8))}.$$

Set

$$\bar{h}_j = \frac{1}{\sqrt{N_j}} \sum_{i \in J_j} \frac{1}{\sqrt{N_j}} f_i = h_j + \frac{1}{\sqrt{N_j}} \sum_{i \in J_j} g_i \text{ (by (9))}.$$

Then

$$\|\bar{h}_j - h_j\| = \left\| \frac{1}{\sqrt{N_j}} \sum_{i \in J_j} g_i \right\|_p = \frac{1}{\sqrt{N_j}} \left( \sum_{i \in J_j} \|g_i\|_p^p \right)^{1/p} \leq N_j^{\frac{1}{p}-\frac{1}{2}} \|f\|_p, \text{ (since } \|g_i\|_p \leq \|f\|_p \text{)}.$$

By (8), for  $\varepsilon$  sufficiently small, it follows that  $(\bar{h}_j)$  is equivalent to  $(h_j)$  and so  $L_p$  embeds into  $X$ .

Let  $E_j = \text{span}\{f_i : i \in J_j\}$  and  $F_j = \text{span}\{\{g_i : i \in J_j\} \cup \{h_j\}\}$ . Since the  $g_i$ 's are disjointly supported and  $(h_i)$  is unconditional, it follows that  $(F_j)$  is an unconditional FDD for its closed linear span,  $Y$ .  $E_j$  is a co-dimension one subspace of  $F_j$  and thus  $(E_j)$  is an unconditional FDD for its closed linear span,  $X$ .  $Y$  is isometric to  $L_p[0, 1] \oplus \ell_p$  and this in turn is isomorphic to  $L_p(\mathbb{R})$ .

Since  $X$  contains an isomorphic copy of  $L_p$ , to prove that  $X$  is isomorphic to  $L_p$  it suffices to prove that  $X$  is complemented in  $Y$ . Indeed  $Y$  is complemented in  $L_p$  and so if  $X$  is complemented in  $Y$  then  $X$  is a complemented subspace of  $L_p$  which contains a complemented copy of  $L_p$  (by [JMST]). By Pełczyński's decomposition method [LT],  $X$  is isomorphic to  $L_p$ .

To accomplish this we will first define certain projections  $P_j$  from  $F_j$  onto  $E_j$  and then prove that  $P = \sum_j P_j$  is a projection of  $Y$  onto  $X$ .

Let  $z_j = \sum_{i \in J_j} N_j^{-1/p} g_i \in F_j \setminus E_j$ . We will prove that the seminormalized sequence  $(z_j)$  satisfies  $d(z_j, E_j) \geq c > 0$  for all  $j$  and some  $c$ .  $P_j$  will then be the projection of  $F_j$  onto  $E_j$  that sends  $z_j$  to 0 and hence the  $P_j$ 's will be uniformly bounded.

We let  $(\tilde{h}_j)$  be the biorthogonal sequence to  $h_j$  in  $L_q[0, 1]$  given by  $\tilde{h}_j = |h_j|^{p-1} \text{sign}(h_j)$ . Thus  $\|\tilde{h}_j\|_q = 1$ . Set  $\tilde{g}_j = \frac{|g_j|^{p-1} \text{sign}(g_j)}{\|g_j\|_p^p} \in L_q[0, 1]$ . Note that  $\tilde{g}_j(g_j) = 1$  and  $\tilde{g}_j(g_i) = 0$  for  $i \neq j$ . Furthermore  $\|\tilde{g}_j\|_q = \frac{1}{\|g_j\|_p}$  and  $\text{supp}(\tilde{g}_j) = \text{supp}(g_j)$ .

Next let

$$\phi_j = N_j^{\frac{1}{2}-\frac{1}{q}} \tilde{h}_j - N_j^{-1/q} \sum_{i \in J_j} \tilde{g}_i .$$

$F_j$  is isometric to  $\ell_p^{N_j+1}$  and

$$\|\phi_j|_{F_j}\| = \sup \left\{ \frac{\left| N_j^{\frac{1}{2}-\frac{1}{p}} a_0 - N_j^{-1/q} \sum_{i \in J_j} a_i \right|}{\left( |a_0|^p + \sum_{i \in J_j} |a_i|^p \|g_i\|_p^p \right)^{1/p}} \right\} ,$$

the “sup” is taken over all nonzero  $(a_i)_0^{N_j} \in \ell_p^{N_j+1}$

$$\|\phi_j|_{F_j}\| = \sup \left\{ \frac{\left| N_j^{\frac{1}{2}-\frac{1}{p}} a_0 - \sum_{i \in J_j} N_j^{-1/q} \|\tilde{g}_i\|_q a_i \|g_i\|_p \right|}{\left( |a_0|^p + \sum_{i \in J_j} |a_i|^p \|g_i\|_p^p \right)^{1/p}} \right\} = \left\| N_j^{\frac{1}{2}-\frac{1}{q}} e_0 + \sum_{i \in J_j} N_j^{-1/q} \|\tilde{g}_i\|_q e_i \right\|_{\ell_q^{N_j+1}} ,$$

where  $(e_i)_0^{N_j}$  is the unit vector basis for  $\ell_q^{N_j+1}$ .

Thus, since  $\tilde{h}_j$  and  $(\tilde{g}_i)_{i \in J_j}$  are disjointly supported in  $L_q(\mathbb{R})$

$$\|\phi_j|_{F_j}\| = \left\| N_j^{\frac{1}{2}-\frac{1}{q}} \tilde{h}_j + \sum_{i \in J_j} N_j^{-1/q} \tilde{g}_i \right\|_{L_q} = \|\phi_j\|_{L_q} .$$

Now for  $i_0 \in J_j$ ,

$$\phi_j(f_{i_0}) = N_j^{\frac{1}{2}-\frac{1}{q}} \tilde{h}_j \left( \frac{1}{\sqrt{N_j}} h_j \right) - N_j^{-1/q} \sum_{i \in J_j} \tilde{g}_i(g_{i_0}) = N_j^{-1/q} - N_j^{-1/q} = 0 .$$

Thus  $\text{Ker } \phi_j|_{F_j} = E_j$ . It follows that

$$d(z_j, E_j) \geq \frac{\phi_j(z_j)}{\|\phi_j\|_q} = \frac{1}{\|\phi_j\|_q} \geq c > 0$$

for some  $c$  and all  $j$ , since  $(\phi_j)$  is seminormalized in  $L_q$ .

We define  $P_j : F_j \rightarrow E_j$  by

$$(10) \quad \lambda z_j + \sum_{i \in J_j} b_i \left( \frac{1}{\sqrt{N_j}} h_j + g_i \right) \mapsto \sum_{i \in J_j} b_i \left( \frac{1}{\sqrt{N_j}} h_j + g_i \right),$$

and let  $P = \sum_{j=1}^{\infty} P_j$ . Let  $C = \sup_j \|P_j\|$ . Let  $x = \sum_{j=1}^{\infty} x_j$ ,  $x_j = a_j h_j + \sum_{i \in J_j} c_i g_i \in F_j$ .

Then by (10),

$$P(x) = \sum_{j=1}^{\infty} P_j(x_j) = \sum_{j=1}^{\infty} \left( a_j h_j + \sum_{i \in J_j} d_i g_i \right)$$

for some sequence  $(d_i)$ . Thus to show that  $P$  is bounded we need only show that for some  $K < \infty$ ,

$$\left\| \sum_{j=1}^{\infty} \sum_{i \in J_j} d_i g_i \right\|_p \leq K \|x\|.$$

Now

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} \sum_{i \in J_j} d_i g_i \right\|_p &= \left( \sum_{j=1}^{\infty} \left\| \sum_{i \in J_j} d_i g_i \right\|_p^p \right)^{1/p} \\ &\leq C \left( \sum_{j=1}^{\infty} \|x_j\|_p^p \right)^{1/p} = C \left( \sum_{j=1}^{\infty} \left( |a_j|^p + \left\| \sum_{i \in J_j} c_i g_i \right\|_p^p \right) \right)^{1/p}. \end{aligned}$$

$(h_j)$  admits a lower  $\ell_p$  estimate since  $p > 2$ . Thus for some  $\bar{K}$ ,

$$\left\| \sum_{j=1}^{\infty} \sum_{i \in J_j} d_i g_i \right\|_p \leq C \left[ \bar{K}^p \left\| \sum_{j=1}^{\infty} a_j h_j \right\|_p^p + \sum_{j=1}^{\infty} \left\| \sum_{i \in J_j} c_i g_i \right\|_p^p \right]^{1/p} \leq C \bar{K} \|x\|.$$

□

## 5. COMPACTNESS OF RESTRICTION OPERATORS

In the case where  $(T_{\lambda_i} f)_{i=1}^{\infty}$  is an unconditional basic sequence of translates of some  $f \in L_p(\mathbb{R})$ ,  $1 \leq p \leq 2$ , the space  $X_p(f, (\lambda_i))$  must be quite thin as the next proposition reveals.

**Proposition 5.1.** *Let  $f \in L_p(\mathbb{R})$ ,  $1 \leq p \leq 2$ . Let  $T_{\lambda_i} f \equiv f_i$  be such that  $(f_i)$  is unconditional basic. Let  $I \subseteq \mathbb{R}$  be a bounded interval and  $X = X_p(f, (\lambda_i))$ . Then the map  $T_I : X \rightarrow L_p(I)$ ,  $x \mapsto x|_I$ , is a compact operator.*

*Proof.* For  $p = 1$  this follows by the proof of Corollary 2.4 [OSSZ]. In fact this holds under the assumption that  $(f_i)$  is basic (and even less).

Suppose that  $1 < p \leq 2$  and  $\varepsilon > 0$ . Since  $\sum_{i=1}^{\infty} \|f_i|_I\|_p^p < \infty$  (see Proposition 2.1, [OSSZ]) there exists  $N \in \mathbb{N}$  so that  $(\sum_{i=N}^{\infty} \|f_i|_I\|^p)^{1/p} < \varepsilon$ . Let  $x = \sum_{i=N}^{\infty} a_i f_i$ ,  $\|x\|_p = 1$ . Then

$$\|x|_I\| \leq \sum_{i=N}^{\infty} |a_i| \|f_i|_I\| \leq \left( \sum_{i=N}^{\infty} |a_i|^q \right)^{1/q} \left( \sum_{i=N}^{\infty} \|f_i|_I\|_p^p \right)^{1/p}$$

by Hölder's inequality ( $\frac{1}{p} + \frac{1}{q} = 1$ ). Since  $q \geq 2$ ,  $(\sum_{i=N}^{\infty} |a_i|^q)^{1/q} \leq (\sum_{i=N}^{\infty} |a_i|^2)^{1/2}$ . Furthermore, by the unconditionality of  $(f_i)$ , there exists a constant  $K$  so that

$$\left( \sum_{i=1}^{\infty} |a_i|^2 \right)^{1/2} \leq K \|x\| = K .$$

$K$  depends only on  $p$ , the unconditionality constant of  $(f_i)$  and  $\|f\| = \|f_i\|$  for  $i \in \mathbb{N}$ . Thus  $\|x|_I\| \leq K\varepsilon$ . This proves that  $T_I$  is a compact operator on  $X$ .  $\square$

We will show in Proposition 5.4 below that Proposition 5.1 fails for  $p > 2$ . However, in the range  $2 < p \leq 4$  we have the following result whose proof can be extracted from the proof of [OSSZ, Theorem 2.11].

**Proposition 5.2.** *Let  $f \in L_p(\mathbb{R})$ ,  $2 < p \leq 4$ . Let  $T_{\lambda_i} f \equiv f_i$  be such that  $(f_i)$  is unconditional basic. Then there is a basic sequence  $(g_i)$  in  $L_p(\mathbb{R})$  equivalent to  $(f_i)$  such that with  $Y = \overline{\text{span}}\{g_i : i \in \mathbb{N}\}$  for any bounded interval  $I \subseteq \mathbb{R}$  the map  $T_I : Y \rightarrow L_p(I)$ ,  $y \mapsto y|_I$ , is a compact operator.*

*Proof.* Let  $(h_j)$  be the normalized Haar basis for  $L_p[0, 1]$ . For  $i \in \mathbb{Z}$  and  $j \in \mathbb{N}$  let  $h_i^j$  be  $h_j$  translated to  $[i, i + 1]$ . Thus  $(h_i^j)$  is a normalized unconditional basis of  $L_p(\mathbb{R})$ .

By approximating each  $f_i$  by a simple dyadic function we find a seminormalized block basis  $(g_i)$  of  $(h_i^j)$  such that

$$(11) \quad \sum_{i=1}^{\infty} \| |f_i| - |g_i| \|_p < \infty .$$



By a very useful observation of Schechtman [S] it follows that  $(f_i)$  is equivalent to  $(g_i)$ .

Set  $Y = \overline{\text{span}}\{g_i : i \in \mathbb{N}\}$  and let  $I$  be a bounded interval. To show that  $T_I : Y \rightarrow L_p(I)$  is compact we can assume that  $I = [-M, M]$  for some  $M \in \mathbb{N}$ . It follows from (11) and [OSSZ, Proposition 2.1] that  $\sum_{i=1}^{\infty} \|g_i|_I\|_p^p < \infty$ . Fix  $\varepsilon > 0$  and choose  $N$  with

$$(12) \quad \sum_{i=N}^{\infty} \|g_i|_I\|_p^p < \varepsilon .$$

We note that  $(g_i|_I)$  is a block basis of  $(h_i^j)_{(j \in \mathbb{N}, -M \leq i < M)}$  (after omitting zero vectors), and thus it is unconditional basic. Let  $y = \sum_{i=N}^{\infty} a_i g_i \in Y$ . Recalling that seminormalized unconditional basic sequences in  $L_p(\mathbb{R})$  satisfy lower  $\ell_p$  and upper  $\ell_2$  estimates, we obtain the following inequalities with some constant  $C$  (dependent only on  $p$  and the norm of  $f$ ).

$$\begin{aligned} \|y|_I\|_p &= \left\| \sum_{i=N}^{\infty} a_i g_i|_I \right\|_p \leq C \left( \sum_{i=N}^{\infty} |a_i|^2 \|g_i|_I\|_p^2 \right)^{1/2} \\ &\leq C \|(a_i)_{i=N}^{\infty}\|_{\ell_p} \left( \sum_{i=N}^{\infty} \|g_i|_I\|_p^{\frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \quad (\text{using Hölder's inequality with } \frac{p}{2} \text{ and } \frac{p}{p-2}) \\ &\leq C^2 \|y\|_p \left( \sum_{i=N}^{\infty} \|g_i|_I\|_p^p \right)^{1/p} \leq C^2 \varepsilon^{1/p} \|y\|_p \quad (\text{using } \frac{2p}{p-2} \geq p) . \end{aligned}$$

This completes the proof.  $\square$

It is worth noting that when the operators  $T_I$  on some subspace  $X \subset L_p(\mathbb{R})$  are compact for all bounded intervals  $I$  then  $X$  must embed into  $\ell_p$  in a natural way as the next proposition reveals. This observation and Proposition 5.1 and 5.2 above simplify some arguments in [OSSZ].

If  $P$  is a partition of  $\mathbb{R}$  into bounded intervals  $(I_j)$  we let  $\mathbb{E}_P$  denote the *conditional expectation operator* on  $L_p(\mathbb{R})$  given by

$$\mathbb{E}_P(f) = \sum_{k=1}^{\infty} \int_{I_k} f(\xi) d\xi \frac{\chi_{I_k}}{m(I_k)} .$$

**Proposition 5.3.** *Let  $X$  be a subspace of  $L_p(\mathbb{R})$ ,  $1 \leq p < \infty$ . If for all bounded intervals  $I \subset \mathbb{R}$  the operator*

$$T_I : X \rightarrow L_p(I), x \mapsto x|_I$$

*is compact, then for all  $\varepsilon > 0$  there exist a partition  $P$  of  $\mathbb{R}$  into bounded intervals so that for all  $x \in S_X$ ,  $\|x - \mathbb{E}_P(x)\| < \varepsilon$ . Thus  $X$  embeds into  $\ell_p$ .*

*Proof.* For  $n \in \mathbb{N}$  let  $Q_n$  be the set of dyadic intervals of length  $2^{-n}$  in  $[0, 1)$ , i.e.

$$Q_n = \{[0, 2^{-n}), [2^{-n}, 2^{1-n}), \dots, [1 - 2^{-n}, 1)\}.$$

Then  $\mathbb{E}_{Q_n}$  converges pointwise to the identity on  $L_p[0, 1]$  and therefore there exists for every relatively compact set  $K \subset L_p[0, 1)$  and every  $\delta > 0$  a large enough  $k \in \mathbb{N}$  so that for all  $x \in K$ ,  $\|x - \mathbb{E}_{Q_k}(x)\| < \varepsilon$ . Choose a sequence  $(\varepsilon_n) \subset (0, 1)$ , with  $\sum \varepsilon_n < \varepsilon$  and for each  $n$  choose a dyadic partition  $P_n$  of the interval  $[n, n+1)$  so that for all  $x \in S_X$ ,  $\|x|_{[n, n+1)} - \mathbb{E}_{P_n}(x|_{[n, n+1)})\| \leq \varepsilon_n$ .

By taking  $P$  to be the union of all  $P_n$  we deduce our claim.  $\square$

Proposition 5.1 fails in the case  $2 < p \leq 4$ , and of course for  $p > 4$  as well, as shown by the next proposition.

**Proposition 5.4.** *Let  $2 < p < \infty$ . There exists  $f \in L_p(\mathbb{R})$  and  $(\lambda_i)_{i=1}^\infty \subseteq \mathbb{N}$  so that for  $f_i = T_{-\lambda_i}f$ ,  $(f_i)_{i=1}^\infty$  is equivalent to the unit vector basis of  $\ell_p$ . Moreover, letting  $I = [0, 1]$  and  $T_I : X_p(f, (-\lambda_i)) \rightarrow L_p(I)$ ,  $x \mapsto x|_I$ ,  $T_I$  is not a compact operator.*

*Proof.* Let  $\frac{1}{p} + \frac{1}{q} = 1$  and let  $(N_j)_{j=1}^\infty$  be a subsequence of  $\mathbb{N}$  satisfying  $\sum_{j=1}^\infty N_j^{q-p} < \infty$ . Set  $m_j = \lceil N_j^q \rceil$  for  $j \in \mathbb{N}$  and let  $(x_j)_{j=1}^\infty$  be a normalized sequence of disjointly supported elements in  $L_p(I)$ . Let  $(J_j)_{j=1}^\infty$  be a partition of  $\mathbb{N}$  into successive intervals so that  $|J_j| = m_j$  for all  $j$ .

For  $i \in J_j$ , let  $x_i^j$  be  $x_j$  placed on the interval  $[3^i, 3^i + 1]$  by right translation of  $3^i$  units.

Define

$$f = \sum_{j=1}^{\infty} \sum_{i \in J_j} \frac{1}{N_j} x_i^j .$$

Note that

$$\|f\|_p^p = \left\| \sum_{j=1}^{\infty} \sum_{i \in J_j} \frac{1}{N_j} x_i^j \right\|_p^p = \sum_{j=1}^{\infty} m_j \frac{1}{N_j^p} \leq 2 \sum_{j=1}^{\infty} \frac{N_j^q}{N_j^p} < \infty$$

so  $f \in L_p(\mathbb{R})$ .

Setting  $f_i = T_{-3^i} f$ , for  $i \in \mathbb{N}$ , we have, as in the proof of Theorem 4.1,

$$(13) \quad f_i = \frac{1}{N_j} x_j + g_i , \quad \text{for } i \in J_j ,$$

where the  $g_i$ 's are disjointly supported, seminormalized and with supports disjoint from  $I$ .  $(g_i)$  is thus equivalent to the unit vector basis of  $\ell_p$ .

To see that  $(f_i)$  is equivalent to the unit vector basis of  $\ell_p$ , by (13) it is sufficient to prove that for all  $(a_i)_{i=1}^{\infty} \in \ell_p$ ,

$$(14) \quad \left\| \sum_{i=1}^{\infty} a_i f_i|_I \right\|_p \leq 2 \left( \sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} .$$

First note that for  $j \in \mathbb{N}$ ,

$$\frac{1}{N_j} \left| \sum_{i \in J_j} a_i \right| \leq \frac{1}{N_j} \left( \sum_{i \in J_j} |a_i|^p \right)^{1/p} m_j^{1/q} \leq 2 \left( \sum_{i \in J_j} |a_i|^p \right)^{1/p} .$$

Hence

$$\left\| \sum_{i=1}^{\infty} a_i f_i|_I \right\|_p^p = \left\| \sum_{j=1}^{\infty} \sum_{i \in J_j} a_i \frac{1}{N_j} x_j \right\|_p^p = \sum_{j=1}^{\infty} \left| \sum_{i \in J_j} a_i \frac{1}{N_j} \right|^p \leq 2 \sum_{i=1}^{\infty} |a_i|^p ,$$

which proves (14).

To see that  $T_I$  is not compact, define  $y_j = \sum_{i \in J_j} f_i$ .  $\|y_j\|$  is of the order  $m_j^{1/p}$  and  $\|y_j|_I\| = \left\| \sum_{i \in J_j} \frac{1}{N_j} x_j \right\| = \frac{m_j}{N_j} \geq m_j^{1/p}$ . Thus  $m_j^{-1/p} y_j$  is seminormalized and weakly null in  $L_p(\mathbb{R})$ , but  $\|T_I m_j^{-1/p} y_j\|_p \geq 1$  for all  $j$ .  $\square$

Using much the same argument we have

**Proposition 5.5.** *Let  $2 < p < \infty$ . There exists  $f \in L_p(\mathbb{R})$  and translations of  $f$ ,  $f_i = T_{-3^i}f$ ,  $i \in \mathbb{N}$ , so that*

- i)  $(f_i)$  is basic,
- ii)  $L_p(\mathbb{R})$  embeds isomorphically into  $X_p(f, (-3^i))$ ,
- iii)  $(f_i)$  can be blocked into an unconditional FDD.

*Sketch.* Let  $(h_j)$  be the normalized Haar basis for  $L_p[0, 1]$ . For  $i, j \in \mathbb{N}$ , let  $h_i^j$  be  $h_j$  translated to  $[3^i, 3^i + 1]$ . Set  $f = \sum_j \sum_{i \in J_j} \frac{1}{N_j} h_i^j$  where  $|J_j| = m_j \equiv \lceil N_j^q \rceil$  and  $(J_j)$  partitions  $\mathbb{N}$  into successive intervals. As above  $f_i = \frac{1}{N_j} h_j + g_i$ , for  $i \in J_j$ , where  $(g_i)$  is seminormalized and disjointly supported in  $\mathbb{R} \setminus [0, 1]$ .

If  $y_j = \sum_{i \in J_j} f_i$ , it follows that  $m_j^{-1/p} y_j = h_j + e_j$  where  $(e_j)$  is seminormalized and disjointly supported in  $\mathbb{R} \setminus [0, 1]$ . Since  $(h_i)$  admits a lower  $\ell_p$ -estimate, it follows that  $(h_j + e_j)$  is equivalent to  $(h_j)$ , proving ii).

Set  $F_j = \text{span}\{f_i : i \in J_j\}$  and note that  $F_j \subseteq \overline{F_j} = \text{span}\{h_j, (g_i)_{i \in J_j}\}$ . Since  $(\overline{F_j})$  is an unconditional FDD, so is  $(F_j)$ .

To see that  $(f_i)$  is basic we need only note that  $(f_i)_{i \in J_j}$  is uniformly equivalent, over  $j$ , to the unit vector basis of  $\ell_p^{m_j}$ , as demonstrated in the proof of Proposition 5.4.  $\square$

## 6. OPEN PROBLEMS

We end with a collection of remaining open problems.

**Problems 6.1.** Let  $f \in L_p(\mathbb{R})$  and let  $(f_i)$  be a sequence of translates of  $f$ .

- i) For  $1 < p < \infty$ , can  $(f_i)$  ever be a basis of  $L_p(\mathbb{R})$ ?
- ii) For  $1 < p < 2$ , can  $(f_i)$  ever be basic such that  $L_p$  embeds into  $\overline{\text{span}(f_i)}$ ?
- iii) For  $1 < p < \infty$ , can  $(f_i)$  ever be blocked into an (unconditional) FDD for  $L_p(\mathbb{R})$ ?

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